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Twisted Galilean symmetry and the Pauli principle at low energies

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Abstract

We show the twisted Galilean invariance of the noncommutative parameter, even in the presence of spacetime noncommutativity. We then obtain the deformed algebra of the Schrödinger field in configuration and momentum space by studying the action of the twisted Galilean group on the non-relativistic limit of the Klein–Gordon field. Using this deformed algebra we compute the two-particle correlation function to study the possible extent to which the previously proposed violation of the Pauli principle may impact at low energies. It is concluded that any possible effect is probably well beyond detection at current energies.

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1. Introduction

The studies of noncommutative (NC) geometry and its implications have gained considerable importance in recent times as these studies are motivated both from string theory [1] and from certain condensed matter systems such as the quantum Hall effect [2–4]. Here the canonical NC structure is given by the following operator-valued spacetime coordinates,

$$\left[x^{\mu}_{op}, x^{\nu}_{op}\right] = \mathrm{i}\theta^{\mu\nu}. \tag{1.1}$$

Instead of working with functions of these operator-valued coordinates, one can alternatively work with functions of c-numbered coordinates provided one composes the functions through the $\star$ product defined as [5]

$$\begin{aligned} \alpha *_{\theta} \beta(x) &= \left[\alpha \exp\left(\frac{\mathrm{i}}{2}\overleftarrow{\partial}_{\mu}\theta^{\mu\nu}\overrightarrow{\partial}_{\nu}\right)\beta\right](x), \\ \theta^{\mu\nu} &= -\theta^{\nu\mu} \in \mathbb{R}, \qquad x = (x^0, x^1, \ldots, x^d). \end{aligned} \tag{1.2}$$

The Poincaré group $\mathcal{P}$ or the diffeomorphism group $\mathcal{D}$ which acts on the NC spacetime $\mathbb{R}^{d+1}$ defines a natural action on smooth functions $\alpha \in C^{\infty}(\mathbb{R}^{d+1})$ as

$$(g\alpha)(x) = \alpha(g^{-1}x), \tag{1.3}$$

for $g \in \mathcal{P}$ or $\mathcal{D}$. However, in general

$$(g\alpha) *_\theta (g\beta) \neq g(\alpha *_\theta \beta), \tag{1.4}$$

showing that the action of the group $\mathcal{P}$ or $\mathcal{D}$ is not an automorphism of the algebra $\mathcal{A}_\theta(\mathbb{R}^{d+1})$, unless one considers the translational subgroup. This violation of the Poincaré symmetry in particular is accompanied by the violation of microcausality, spin statistics and the CPT theorem in general [5, 7]. These results, which follow from the basic axioms in the canonical (commutative) quantum field theory (QFT), are no longer satisfied in the presence of noncommutativity in the manner discussed above. Besides, NC field theories are afflicted with infrared/ultraviolet (IR/UV) mixing. It is however possible for some of these results to still go through even after postulating weaker versions of the axioms used in the standard QFT. For example, one can consider the proof of the CPT theorem given by Alvarez-Gaume *et al* [8] where they consider the breaking of the Lorentz symmetry down to the subgroup $O(1, 1) \times SO(2)$, and replace the usual causal structure, given by the light cone, by the light-wedge associated with the $O(1, 1)$ factor of the kinematical symmetry group. One can also consider the derivation of CPT and spin-statistics theorems by Franco *et al* [9] where they invoke only 'asymptotic commutativity', i.e. assuming that the fields to be commuting at sufficiently large spatial separations.

As all these problems basically stemmed from the above-mentioned non-invariance (1.4), it is desirable to look for some way to restore the invariance. Indeed, as has been discovered by Chaichian *et al* [6] and Dimitrijevic *et al* [10, 11] (see also the prior work of Oeckl [12]), this invariance is restored by introducing a deformed coproduct, thereby modifying the corresponding Hopf algebra. Since then, this deformed or twisted coproduct has been used extensively in the framework of relativistic quantum field theory, as this approach seems to be quite promising.

Two interesting consequences follow from the twisted implementation of the Poincaré group. The first is that there is apparently no longer any IR/UV mixing [13], suggesting that the high and low energy sectors decouple, in contrast to the untwisted formulation. The second striking consequence is an apparent violation of Pauli's principle [14]. This seems to be unavoidable if one wants to restore the Poincaré invariance through the twisted coproduct. If there is no IR/UV mixing, one would expect that any violation of Pauli's principle would impact in either the high or low energy sector. Experimental observation at present energies seems to rule out any effect at low energies; therefore, if this picture is a true description of nature, we expect that any violation of the Pauli principle can only appear at high energies. It does, therefore, seem worthwhile as a consistency check to investigate this question in more detail and to establish precisely what the possible impact may be at low energies and why it may not be observable. One of the quantities where spin statistics manifests itself very explicitly is the two-particle correlation function. A way of addressing this issue would therefore be to study the low temperature limit of the two-particle correlation function in a twisted implementation of the Poincaré group. Since we are at low energies it would, however, be sufficient to study the non-relativistic limit, i.e. the Galilean symmetry. We therefore need to consider the question of whether the Galilean symmetry can also be restored by a suitable twist of the coproduct. This is a non-trivial point that requires careful investigation, as the Galilean algebra admits a central extension, in the form of mass, unlike the Poincaré case, and the boost generator does not have a well-defined coproduct action. It may be recalled, in this context, that the presence of spacetime noncommutativity spoils the NC structure under

Galileo boost. This question is all the more important because of the observation made by Bahns *et at* [15] that the presence of spacetime noncommutativity does not spoil the unitarity of the NC theory. However, we show that the presence of spacetime noncommutativity in the relativistic case does not have a well-defined non-relativistic ($c \to \infty$) limit. Furthermore, spacetime noncommutativity gives rise to certain operator ordering ambiguities rendering the extraction of a non-relativistic field in the $c \to \infty$ limit non-trivial.

This paper is organized as follows. We discuss mathematical preliminaries introducing the concept of Hopf algebra and the deformed or twisted coproduct in section 2. Section 3 deals with a brief review of the twisted Lorentz transformation properties of quantum spacetime in subsection 3.1, as was discussed in [6, 16]. This is then extended to the non-relativistic case in subsection 3.2. We then discuss briefly the non-relativistic reduction of the Klein–Gordon field to the Schrödinger field in (2 + 1) dimensions in commutative space in section 4, which is then used to obtain the action of the twisted Galilean transformation on the Fourier coefficients in section 5. We eventually obtain the action of the twisted Galilean transformation on the non-relativistic Schrödinger fields in section 6. In section 7 we discuss the implications of the subsequent deformed commutation relations on the two-particle correlation function of a free gas in two spatial dimensions. We conclude in section 8. Finally, we have added an appendix where we have included some important aspects of the Wigner-Inönu group contraction in this context (i.e. Poincaré → Galileo), which we have made use of in the main text.

2. Mathematical preliminaries

In this section we give a brief review of the essential results in [14] for the purpose of application in later sections.

Suppose that a group G acts on a complex vector space V by a representation ρ. We denote this action by

$$v \to \rho(g)v \tag{2.1}$$

for $g \in G$ and $v \in V$. Then the group algebra G^* also acts on V. A typical element of G^* is

$$\int \mathrm{d}g\, \alpha(g)g, \qquad \alpha(g) \in \mathbb{C}, \tag{2.2}$$

where $\mathrm{d}g$ is an invariant measure on G. Its action is

$$v \to \int \mathrm{d}g\, \alpha(g)\rho(g)v. \tag{2.3}$$

Both G and G^* act on $V \otimes_{\mathbb{C}} V$, the tensor product of V's over $\mathbb{C}$, as well. These actions are usually taken to be

$$v_1 \otimes v_2 \to [\rho(g) \otimes \rho(g)]\,(v_1 \otimes v_2) = \rho(g)v_1 \otimes \rho(g)v_2 \tag{2.4}$$

and

$$v_1 \otimes v_2 \to \int \mathrm{d}g\, \alpha(g)\rho(g)v_1 \otimes \rho(g)v_2, \tag{2.5}$$

respectively, for $v_1, v_2 \in V$.

In Hopf algebra theory [17, 18], the action of G and G^* on tensor products is defined by the coproduct Δ_0, a homomorphism from G^* to $G^* \otimes G^*$, which on restriction to G gives a homomorphism from G to $G^* \otimes G^*$. This restriction specifies Δ_0 on all of G^* by linearity. Hence, if

$$\Delta_0 : g \to \Delta_0(g) \qquad \Delta_0(g_1)\Delta_0(g_2) = \Delta_0(g_1 g_2), \tag{2.6}$$

we have

$$\Delta_0\left(\int \mathrm{d}g\ \alpha(g)g\right) = \int \mathrm{d}g\ \alpha(g)\Delta_0(g). \tag{2.7}$$

Suppose next that V is an algebra $\mathcal{A}$ (over $\mathbb{C}$). As $\mathcal{A}$ is an algebra, we have a rule for taking products of elements of $\mathcal{A}$. This means that there is a multiplication map

$$\begin{aligned} &m : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A} \\ &\alpha\otimes\beta \to m(\alpha\otimes\beta) \end{aligned} \tag{2.8}$$

for $\alpha, \beta \in \mathcal{A}$, the product $\alpha\beta$ being $m(\alpha\otimes\beta)$.

It is now essential that Δ_0 be compatible with m, so that

$$m((\rho\otimes\rho)\Delta_0(g)(\alpha\otimes\beta)) = \rho(g)m(\alpha\otimes\beta). \tag{2.9}$$

In the Moyal plane, the multiplication denoted by the map m_θ is NC and depends on $\theta^{\mu\nu}$. It is defined by[3]

$$m_\theta(\alpha\otimes\beta) = m_0\big(\mathrm{e}^{-\frac{\mathrm{i}}{2}(\mathrm{i}\partial_\mu)\theta^{\mu\nu}\otimes(\mathrm{i}\partial_\nu)}\alpha\otimes\beta\big) = m_0(F_\theta\alpha\otimes\beta), \tag{2.10}$$

where m_0 is the pointwise multiplication of two functions and F_θ is the twist element given by

$$F_\theta = \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^{\mu\nu}P_\mu\otimes P_\nu} = \mathrm{e}^{-\frac{\mathrm{i}}{2}(\mathrm{i}\partial_\mu)\theta^{\mu\nu}\otimes(\mathrm{i}\partial_\nu)}, \qquad P_\mu = \mathrm{i}\partial_\mu. \tag{2.11}$$

The twist element F_θ changes the coproduct to

$$\Delta_0(g) \to \Delta_\theta(g) = \hat{F}_\theta^{-1}\Delta_0(g)\hat{F}_\theta \tag{2.12}$$

in order to maintain compatibility with m_θ, as can be easily checked. In the case of the Poincaré group, if $\exp(\mathrm{i}P\cdot a)$ is a translation,

$$(\rho\otimes\rho)\Delta_\theta(\mathrm{e}^{\mathrm{i}P\cdot a})e_p\otimes e_q = \mathrm{e}^{\mathrm{i}(p+q)\cdot a}e_p\otimes e_q, \qquad (e_p(x) = \mathrm{e}^{-\mathrm{i}p\cdot x}), \tag{2.13}$$

while if Λ is a Lorentz transformation

$$(\rho\otimes\rho)\Delta_\theta(\Lambda)e_p\otimes e_q = \big[\mathrm{e}^{\frac{\mathrm{i}}{2}(\Lambda p)_\mu\theta^{\mu\nu}(\Lambda q)_\nu}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}p_\mu\theta^{\mu\nu}q_\nu}\big]e_{\Lambda p}\otimes e_{\Lambda q}. \tag{2.14}$$

These relations are derived in [14]. Finally, let us mention the action of the coproduct Δ_0 on the elements of a Lie algebra $\mathcal{A}$. The coproduct is defined on $\mathcal{A}$ by

$$\Delta_0(X) = X\otimes 1 + 1\otimes X. \tag{2.15}$$

Its action on the elements of the corresponding universal covering algebra $\mathcal{U}(\mathcal{P})$ can be calculated through the homomorphism [19], i.e.

$$\Delta_0(XY) = \Delta_0(X)\Delta_0(Y) = XY\otimes 1 + X\otimes Y + Y\otimes X + 1\otimes XY. \tag{2.16}$$

One can also easily check that this action of the coproduct on the Lie algebra is consistent with the action on the group element defined by

$$\Delta_0(g) = g\otimes g. \tag{2.17}$$

[3] The signature we are using is $(+, -, -, \ldots)$.

3. Transformation properties of tensors under spacetime transformation

3.1. Lorentz transformation

To set the scene for the rest of the paper, we give a brief review of the Lorentz transformation properties in the commutative case in this subsection. This, as we shall see, turns out to be essential in understanding the action of the Lorentz generators on any vector or tensor field. Consider an infinitesimal Lorentz transformation

$$x^\mu \to x'^\mu = x^\mu + \omega^{\mu\nu} x_\nu, \tag{3.1}$$

where $\omega^{\mu\nu}$ is an infinitesimal constant ($\omega^{\mu\nu} = -\omega^{\nu\mu}$). Under this transformation, any vector field A_μ transforms as

$$A_\mu \to A'_\mu(x') = A_\mu(x) + \omega_\mu{}^\lambda A_\lambda(x). \tag{3.2}$$

Hence the functional change in $A_\mu(x)$ reads

$$\begin{aligned}\delta_0 A_\mu(x) &= A'_\mu(x) - A_\mu(x)\\ &= \omega^{\nu\lambda} x_\nu \partial_\lambda A_\mu(x) + \omega_{\mu\nu} A^\nu\\ &= -\frac{\mathrm{i}}{2}\omega^{\nu\lambda} J_{\nu\lambda} A_\mu,\end{aligned} \tag{3.3}$$

where $J_{\nu\lambda} = M_{\nu\lambda} + S_{\nu\lambda}$ are the total Lorentz generators with $M_{\mu\nu}$ and $S_{\mu\nu}$ identified with orbital and spin parts, respectively. This immediately leads to the representation of $M_{\nu\lambda}$,

$$M_{\nu\lambda} = \mathrm{i}(x_\nu \partial_\lambda - x_\lambda \partial_\nu) = (x_\nu P_\lambda - x_\lambda P_\nu), \qquad P_\lambda = \mathrm{i}\partial_\lambda. \tag{3.4}$$

To find the representation of $S_{\nu\lambda}$, we make use of the relation $\frac{\mathrm{i}}{2}\omega^{\rho\lambda}(S_{\rho\lambda}A)_\mu = \omega_{\mu\nu}A^\nu$ obtained by comparing both sides of (3.3). This leads to

$$(S_{\alpha\beta})_{\mu\nu} = \mathrm{i}(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha}). \tag{3.5}$$

It can now be easily checked that $M_{\mu\nu}$, $S_{\mu\nu}$ and $J_{\mu\nu}$ all satisfy the same homogeneous Lorentz algebra $SO(1,3)$:

$$[M_{\mu\nu}, M_{\lambda\rho}] = \mathrm{i}(\eta_{\mu\lambda}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\lambda} - \eta_{\nu\lambda}M_{\mu\rho} + \eta_{\nu\rho}M_{\mu\lambda}). \tag{3.6}$$

Setting $A_\mu = x_\mu$, where x_μ represents a position coordinate of a spacetime point, yields

$$\delta_0 x_\mu = -\frac{\mathrm{i}}{2} w^{\nu\lambda}(M_{\nu\lambda} + S_{\nu\lambda})x_\mu = 0 \tag{3.7}$$

as expected, since the Lie derivative of the 'radial' vector field $\vec{X} = x^\mu\partial_\mu$ w.r.t. the 'rotation' generators (3.4) $M_{\mu\nu}$ vanishes, i.e. $\mathcal{L}_{M_{\mu\nu}}\vec{X} = 0$.

Now we observe that the change in x_μ (not the functional change $\delta_0 x_\mu$ as in (3.3)) defined by

$$\delta x_\mu = x'_\mu - x_\mu = \omega_\mu{}^\nu x_\nu \tag{3.8}$$

can be identified as the action of $S_{\nu\lambda}$ on x_μ,

$$\delta x_\mu = \omega_\mu{}^\nu x_\nu = -\frac{\mathrm{i}}{2}\omega^{\nu\lambda}(S_{\nu\lambda}x)_\mu, \tag{3.9}$$

with the representation of $S_{\nu\lambda}$ given in (3.5). Using (3.7), one can also obtain the action of $M_{\nu\lambda}$ on x_μ,[4]

$$\delta x_\mu = -\frac{\mathrm{i}}{2}\omega^{\nu\lambda} M_{\nu\lambda} x_\mu. \tag{3.10}$$

[4] Note that $\delta A_\mu = A'_\mu(x') - A_\mu(x) = \omega_\mu{}^\lambda A_\lambda(x)$ is not the functional change and δx_μ in (3.10) is obtained by setting $A_\mu = x_\mu$.

One can generalize this to higher second-rank tensors $f_{\rho\sigma}(x) = x_\rho x_\sigma$ as

$$\delta(x_\lambda x_\sigma) = \left(-\frac{\mathrm{i}}{2} w^{\mu\nu} M_{\mu\nu}\right)(x_\lambda x_\sigma), \tag{3.11}$$

since we can write

$$\begin{aligned} M_{\mu\nu} f_{\rho\sigma} &= \mathrm{i}(x_\mu \partial_\nu - x_\nu \partial_\mu) f_{\rho\sigma} \\ &= \mathrm{i}(f_{\mu\sigma}\eta_{\nu\rho} - f_{\nu\sigma}\eta_{\mu\rho} + f_{\rho\nu}\eta_{\mu\sigma} - f_{\rho\mu}\eta_{\nu\sigma}), \end{aligned} \tag{3.12}$$

where we have made use of (3.4). This indeed shows the covariant nature of the transformation properties of $f_{\rho\sigma}$.

We now review the corresponding covariance property in the NC case under the twisted coproduct of Lorentz generators [6, 16]. The issue of violation of the Lorentz symmetry in noncommutative quantum field theories has been known for a long time, since the field theories defined on a noncommutative spacetime obeying the commutation relation (1.1) between the coordinate operators, where $\theta_{\mu\nu}$ is treated as a constant antisymmetric matrix, are obviously not Lorentz invariant. However, it has been shown [6] that there exists a new kind of twisted Poincaré symmetry under which quantum field theories defined on noncommutative spacetime are still Poincaré invariant.

To generalize to the NC case, first note that the star product between two vectors x_μ and x_ν given as $x_\mu \star x_\nu$ is not symmetric, unlike in the commutative case. One can, however, write this as

$$x_\mu \star x_\nu = x_{\{\mu} \star x_{\nu\}} + \frac{\mathrm{i}}{2}\theta_{\mu\nu}, \tag{3.13}$$

where the curly brackets { } denotes symmetrization in the indices μ and ν. This can be easily generalized to higher ranks, showing that every tensorial object of the form $(x_\mu \star x_\nu \star \cdots \star x_\sigma)$ can be written as a sum of symmetric tensors of equal or lower rank, so that the basis representation is symmetric. Consequently $f_{\rho\sigma}$ should be replaced by the symmetrized expression $f^\theta_{\rho\sigma} = x_{\{\rho} \star x_{\sigma\}} = \frac{1}{2}(x_\rho \star x_\sigma + x_\sigma \star x_\rho)$, and correspondingly the action of the Lorentz generator should be applied through the twisted coproduct (2.12),

$$\begin{aligned} M^\theta_{\mu\nu} f^\theta_{\rho\sigma} &= M^\theta_{\mu\nu} m_\theta (x_\rho \otimes x_\sigma) = m_\theta(\Delta_\theta(M_{\mu\nu})(x_\rho \otimes x_\sigma)) \\ &= \mathrm{i}\big(f^\theta_{\mu\sigma}\eta_{\nu\rho} - f^\theta_{\nu\sigma}\eta_{\mu\rho} + f^\theta_{\rho\nu}\eta_{\mu\sigma} - f^\theta_{\rho\mu}\eta_{\nu\sigma}\big). \end{aligned} \tag{3.14}$$

In the above equation, $M^\theta_{\mu\nu}$ denotes the usual Lorentz generator, but with the action of a twisted coproduct. In [6], it was shown that $M^\theta_{\mu\nu}(\theta^{\rho\sigma}) = 0$, and

$$M^\theta_{\mu\nu}\big(S^2_t\big) = 0; \qquad \big(S^2_t = x_\sigma \star x_\sigma\big), \tag{3.15}$$

i.e. the antisymmetric tensor $\theta^{\rho\sigma}$ is twisted-Poincaré invariant.

3.2. Twisted Galilean invariance

In this subsection we extend the above twisted Poincaré result to the corresponding non-relativistic case. To demonstrate the need for this, consider the Galilean boost transformation,

$$t \to t' = t \qquad x^i \to x'^i = x^i - v^i t, \tag{3.16}$$

applied in the NC Galilean spacetime having the following NC structure:

$$[t, x^i] = \mathrm{i}\theta^{0i}; \qquad [x^i, x^j] = \mathrm{i}\theta^{ij}. \tag{3.17}$$

The corresponding expression in the boosted frame is given by

$$\begin{aligned} &[t', x'^i] = [t, x^i] = \mathrm{i}\theta^{0i} \\ &[x'^i, x'^j] = \mathrm{i}\theta^{ij} + \mathrm{i}(\theta^{0i}v^j - \theta^{0j}v^i). \end{aligned} \tag{3.18}$$

This shows that the NC structure in the primed frame does not preserve its structure unless spacetime noncommutativity disappears, i.e. $\theta^{0i} = 0$. In this section we demonstrate that even in the presence of spacetime noncommutativity the Galilean symmetry can be restored through an appropriate twist. To do this we consider a tangent vector field $\vec{A}(x) = A^\mu(x)\partial_\mu$ in Galilean spacetime. Under Galilean transformations (3.16),

$$\begin{aligned} A^i(x) &\to A'^i(x') = \frac{\partial x'^i}{\partial x^\mu} A^\mu(x) = A^i(x) - v^i A^0(x) \\ A^0(x) &\to A'^0(x') = A^0(x). \end{aligned} \tag{3.19}$$

From (3.19) it follows that

$$\begin{aligned} \delta_0 A^\mu(x) &= A'^\mu(x) - A^\mu(x) \\ &= \mathrm{i}v^j\left(-\mathrm{i}t\partial_j A^\mu(x) + \mathrm{i}\delta^\mu_j A^0(x)\right) \\ &= \mathrm{i}v^j K_j A^\mu(x), \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} K_j A^\mu(x) &= \left(-\mathrm{i}t\partial_j A^\mu(x) + \mathrm{i}\delta^\mu_j A^0(x)\right) \\ &= -tP_j A^\mu(x) + \mathrm{i}\delta^\mu_j A^0(x). \end{aligned} \tag{3.21}$$

Setting $A^\mu(x) = x^\mu$,[5] we easily see that $K_j x^\mu = 0$, from which we get

$$\delta x^\mu = \mathrm{i}v^j t P_j x^\mu = \mathrm{i}v^j K^{(0)}_j x^\mu, \tag{3.22}$$

where $K^{(0)}_j = tP_j$. This is the counterpart of (3.10) in the Galilean case. In other words, here $K^{(0)}_j$ plays the same role as $M_{\mu\nu}$ in the relativistic case. Indeed, one can check that at the commutative level it has its own coproduct action

$$K^{(0)}_j m(x^\mu \otimes x^\nu) = m\left(\Delta_0\left(K^{(0)}_j\right)(x^\mu \otimes x^\nu)\right). \tag{3.23}$$

Here $K^{(0)}_j$ is clearly the boost generator $K^{(M)}_j$ (see equation A.8 in the appendix) with $M = 0$. Note that with $M \neq 0$, $K^{(M)}_j$ does not have the right coproduct action (3.23). This is also quite satisfactory from the point of view that the noncommutativity of spacetime is an intrinsic property and should have no bearing on the mass of the system inhabiting it. We also point out another dissimilarity between the relativistic and non-relativistic case. In the relativistic case, the generators $M_{\mu\nu}$ (3.4) can be regarded as the vector field whose integral curve generates the Rindler trajectories, i.e. the spacetime trajectories of a uniformly accelerated particle. On the other hand, the vector field associated with the parabolic trajectories of a uniformly accelerated particle in the non-relativistic case is given by K^{NR}_i (A.5), which however cannot be identified with the Galileo boost generator $K^{(M)}_j$ (A.8) (see the appendix), unlike $M_{\mu\nu}$ in the relativistic case.

At the NC level the action of the Galilean generator should be applied through the twisted coproduct

$$K^{\theta(0)}_j m_\theta(x^\mu \otimes x^\nu) = m_\theta\left(\Delta_\theta\left(K^{(0)}_j\right)(x^\mu \otimes x^\nu)\right). \tag{3.24}$$

Using this and noting $\Delta_\theta\left(K^{(0)}_j\right) = \Delta_0\left(K^{(0)}_j\right)$ we have

$$\begin{aligned} &K^{\theta(0)}_j m_\theta(x^\mu \otimes x^\nu) = \mathrm{i}t\left(x^\mu\delta^\nu_j + \delta^\mu_j x^\nu\right) \\ \Rightarrow\quad &K^{\theta(0)}_j m_\theta(x^\mu \otimes x^\nu - x^\nu \otimes x^\mu) = 0 \\ \Rightarrow\quad &K^{\theta(0)}_j(\theta^{\mu\nu}) = 0, \end{aligned} \tag{3.25}$$

[5] Here we identify x^0 to be just the time t, rather than ct.

i.e. the antisymmetric tensor $\theta^{\mu\nu}$ is invariant under the twisted Galilean boost. Since the rest of the Galileo generators have the same form as that of the Poincaré generators, discussed in the previous section, this shows the complete twisted Galilean invariance of $\theta^{\mu\nu}$.

4. Non-relativistic reduction in commutative space

In this section we discuss the non-relativistic reduction ($c \to \infty$) of the Klein–Gordon field to the Schrödinger field in 2+1 dimensions[6], as this is used in the subsequent sections to derive the deformed algebra of the Schrödinger field both in the momentum and in the configuration space. The deformed algebra in the momentum space for the Klein–Gordon field has already been derived in [14]. Therefore it is advantageous to consider the non-relativistic limit of such a deformed algebra.

To facilitate the process of constructing the $c \to \infty$ limit, we reintroduce the speed of light 'c' in appropriate places from dimensional consideration, but we still work in the unit $\hbar = 1$. We consider the complex Klein–Gordon field, satisfying the Klein–Gordon equation

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2 + m^2c^2\right)\phi(x) = 0. \tag{4.1}$$

This follows from the extremum condition of the Klein–Gordon action,

$$S = \int \mathrm{d}t\, \mathrm{d}^2\mathbf{x} \left[\frac{1}{c^2}\dot{\phi}^\star\dot{\phi} - \phi'^\star\phi' - c^2m^2\phi^\star\phi\right]. \tag{4.2}$$

The Schrödinger field is identified from the Klein–Gordon field by isolating the exponential factor involving rest mass energy and eventually taking the limit $c \to \infty$. To that end we set

$$\phi(\vec{x}, t) = \frac{\mathrm{e}^{-\mathrm{i}mc^2t}}{\sqrt{2m}}\psi(\vec{x}, t), \tag{4.3}$$

which yields from (4.1) the equation

$$-\frac{1}{2m}\nabla^2\psi = \mathrm{i}\frac{\partial\psi}{\partial t} - \frac{1}{2mc^2}\frac{\partial^2\psi}{\partial t^2}. \tag{4.4}$$

This reduces to the Schrödinger equation of a free positive energy particle in the limit $c \to \infty$. In this limit the action (4.2) also yields the corresponding non-relativistic action as

$$S_{\mathrm{NR}} = \int \mathrm{d}t\, \mathrm{d}^2x\, \psi^\star\left(\mathrm{i}\partial_0 + \frac{1}{2m}\nabla^2\right)\psi. \tag{4.5}$$

The complex scalar field $\phi(\mathbf{x})$ can be Fourier expanded as

$$\phi(\vec{x}, t) = \int \mathrm{d}\mu(k) c[a(k)e_k + b^\dagger(k)e_{-k}], \tag{4.6}$$

where $\mathrm{d}\mu(k) = \frac{\mathrm{d}^2\vec{k}}{2k_0(2\pi)^2}$ is the Lorentz invariant measure and $e_k = \mathrm{e}^{-\mathrm{i}k\cdot x} = \mathrm{e}^{-\mathrm{i}(Et-\vec{k}\cdot\vec{x})}$. The well-known equal time commutation relations between ϕ and Π_ϕ can now be used to find the commutation relation between a_k and $a_k^\dagger$,[7]

$$[a(k), a^\dagger(k')] = (2\pi)^2\frac{2k_0}{c}\delta^2(\vec{k} - \vec{k}'), \tag{4.7}$$

[6] The procedure of non-relativistic reduction holds for any spacetime dimension.

[7] Note $k^\mu = (\frac{E}{c}, \vec{k})$.

and likewise for $b(k)$. In order to get the Fourier expansion of the field in the non-relativistic case, we substitute (4.3) in (4.6), which in the limit $c \to \infty$ yields

$$\psi(\vec{x}, t) = \int \frac{\mathrm{d}^2\vec{k}}{(2\pi)^2} \frac{\tilde{c}(k)}{\sqrt{2m}} \tilde{e}_k = \int \frac{\mathrm{d}^2\vec{k}}{(2\pi)^2} c(k) \tilde{e}_k, \tag{4.8}$$

where $\tilde{e}_k = \mathrm{e}^{-\mathrm{i}\frac{|\vec{k}|^2 t}{2m}} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}$, $\tilde{c}(k) = \lim_{c\to\infty} a(k)$ and $c(k) = \frac{1}{\sqrt{2m}}\tilde{c}(k)$ are the Schrödinger modes. As in (4.4) only the positive energy part survives in the $c \to \infty$ limit, so this limit effectively projects the positive frequency part. The commutation relation (4.7) reduces in the non-relativistic limit ($c \to \infty$) to

$$\begin{aligned} [\tilde{c}(k), \tilde{c}^\dagger(k')] &= (2\pi)^2 2m\delta^2(\vec{k} - \vec{k}') \\ [c(k), c^\dagger(k')] &= (2\pi)^2 \delta^2(\vec{k} - \vec{k}'). \end{aligned} \tag{4.9}$$

From (4.8) and (4.9), we obtain

$$[\psi(\vec{x}, t), \psi^\dagger(\vec{y}, t)] = \delta^2(\vec{x} - \vec{y}). \tag{4.10}$$

5. Action of twisted Galilean transformation on Fourier coefficients

Let us consider the Fourier expansion of the relativistic scalar field $\phi(\vec{x}, t)$,

$$\phi(\vec{x}, t) = \int \mathrm{d}\mu(k) c\tilde{\phi}(k) e_k, \tag{5.1}$$

where we have deliberately suppressed the negative frequency part as it does not survive in the non-relativistic limit $c \to \infty$, as we have seen in the previous section. Considering the action of the Poincaré group elements on ϕ, we get

$$\rho(\Lambda_c)\phi = \int \mathrm{d}\mu(k) c\tilde{\phi}(k) e_{\Lambda_c k} = \int \mathrm{d}\mu(k) c\tilde{\phi}\left(\Lambda_c^{-1} k\right) e_k \tag{5.2}$$

$$\rho(\mathrm{e}^{\mathrm{i}P\cdot a})\phi = \int \mathrm{d}\mu(k) c\, \mathrm{e}^{\mathrm{i}k\cdot a} \tilde{\phi}(k) e_k. \tag{5.3}$$

Thus the representation $\tilde{\rho}$ of the Poincaré group on $\tilde{\phi}(k)$ is specified by

$$\begin{aligned} (\tilde{\rho}(\Lambda_c)\tilde{\phi})(k) &= \tilde{\phi}\left(\Lambda_c^{-1} k\right) \\ (\tilde{\rho}(\mathrm{e}^{\mathrm{i}P\cdot a})\tilde{\phi})(k) &= \mathrm{e}^{\mathrm{i}k\cdot a}\tilde{\phi}(k). \end{aligned} \tag{5.4}$$

Here homogeneous Lorentz transformations have been labelled by Λ_c. The corresponding Galilean transformations will be labelled by Λ_∞ in the $c \to \infty$ limit.

If χ is another scalar field, with Fourier expansion given by

$$\chi(\vec{x}, t) = \int \mathrm{d}\mu(q) c\tilde{\chi}(q) e_q, \tag{5.5}$$

the tensor product of the fields ϕ and χ is given by

$$\phi \otimes \chi = \int \mathrm{d}\mu(k)\, \mathrm{d}\mu(q) c^2 \tilde{\phi}(k) \tilde{\chi}(q) e_k \otimes e_q. \tag{5.6}$$

Using (2.14) one obtains the action of the twisted Lorentz transformation on the above tensor product of the fields

$$\Delta_\theta(\Lambda_c)(\phi \otimes \chi) = \int \mathrm{d}\mu(k)\, \mathrm{d}\mu(q) c^2 \tilde{\phi}\left(\Lambda_c^{-1} k\right) \tilde{\chi}\left(\Lambda_c^{-1} q\right) \mathrm{e}^{\frac{\mathrm{i}}{2} k_\mu \theta^{\mu\nu} q_\nu}\, \mathrm{e}^{-\frac{\mathrm{i}}{2} (\Lambda_c^{-1} k)_\alpha \theta^{\alpha\beta} (\Lambda_c^{-1} q)_\beta} (e_k \otimes e_q). \tag{5.7}$$

Substituting (4.3) in the above equation, one can write the corresponding action of the twisted Lorentz transformations on the tensor product of the fields ψ and ξ (here ξ is the counterpart of ψ for the field χ as in (4.3)) as

$$\Delta_\theta(\Lambda_c)(\psi\otimes\xi) = \int \mathrm{d}\mu(k)\,\mathrm{d}\mu(q) 2mc^2\tilde{\phi}\big(\Lambda_c^{-1}k\big)\tilde{\chi}\big(\Lambda_c^{-1}q\big) \\ \times \mathrm{e}^{\frac{\mathrm{i}}{2}k_i\theta^{ij}q_j}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}(\Lambda_c^{-1}k)_l\theta^{\ln}(\Lambda_c^{-1}q)_n}\,\mathrm{e}^{-2\mathrm{i}O(\frac{1}{c^2},\ldots)}(\tilde{e}_k\otimes\tilde{e}_q). \tag{5.8}$$

Note that we have set $\theta^{0i}=0$ on the right-hand side of the above equation. The underlying reason is that the substitution (4.3) can be carried out only in the absence of spacetime noncommutativity ($\theta^{0i}=0$) as this removes any operator ordering ambiguities in (4.3). This should not, however, be regarded as a serious restriction as theories with spacetime noncommutativity do not represent a low energy limit of string theory [7, 21].

Hence in the limit $c\to\infty$, we can deduce the action of the twisted Galilean transformations (Λ_∞) on tensor products of the non-relativistic fields:

$$\Delta_\theta(\Lambda_\infty)(\psi\otimes\xi) = \int \frac{\mathrm{d}^2\vec{k}\,\mathrm{d}^2\vec{q}}{(2\pi)^4}\tilde{\psi}\big(\Lambda_\infty^{-1}k\big)\tilde{\xi}\big(\Lambda_\infty^{-1}q\big)\,\mathrm{e}^{\frac{\mathrm{i}}{2}mv_1\theta(k_2-q_2)}(\tilde{e}_k\otimes\tilde{e}_q). \tag{5.9}$$

Here we have considered a boost along the x^1 direction with velocity v_1 and $\tilde{\psi}(k) = \lim_{c\to\infty}\tilde{\phi}(k)$, $\tilde{\xi}(q) = \lim_{c\to\infty}\tilde{\chi}(q)$.

From the above, one deduces the action of the twisted Galilean transformations (Λ_∞) on the Fourier coefficients of the non-relativistic fields

$$\Delta_\theta(\Lambda_\infty)(\tilde{\psi}\otimes\tilde{\xi})(k,q) = \tilde{\psi}\big(\Lambda_\infty^{-1}k\big)\tilde{\xi}\big(\Lambda_\infty^{-1}q\big)\,\mathrm{e}^{\frac{\mathrm{i}}{2}mv_1\theta(k_2-q_2)}. \tag{5.10}$$

One can now easily generalize the above result for the case of any arbitary direction of boost as

$$\Delta_\theta(\Lambda_\infty)(\tilde{\psi}\otimes\tilde{\xi})(k,q) = \tilde{\psi}\big(\Lambda_\infty^{-1}k\big)\tilde{\xi}\big(\Lambda_\infty^{-1}q\big)\,\mathrm{e}^{\frac{\mathrm{i}}{2}m\theta\vec{v}\times(\vec{k}-\vec{q})}. \tag{5.11}$$

6. Quantum fields

In this section, we discuss the action of twisted Galilean transformation on non-relativistic Schrödinger fields. A free relativistic complex quantum field ϕ of mass m can be expanded in the NC plane (suppressing the negative frequency part) as

$$\phi(\vec{x},t) = \int \mathrm{d}\mu(k)cd(k)e_k. \tag{6.1}$$

This is just the counterpart of (4.6) where $a(k)$ has been replaced by $d(k)$.[8]

The deformation algebra involving $d(k)$ has already been derived in [14]. In this paper, we derive the deformation algebra for the non-relativistic case. The non-relativistic limit of the complex Klein–Gordon field has already been discussed in the earlier section and the expansion is the following:

$$\psi(\vec{x},t) = \int \frac{\mathrm{d}^2\vec{k}}{(2\pi)^2}\frac{\tilde{u}(k)}{\sqrt{2m}}\tilde{e}_k = \int \frac{\mathrm{d}^2\vec{k}}{(2\pi)^2}u(k)\tilde{e}_k; \qquad u(k) = \frac{1}{\sqrt{2m}}\tilde{u}(k), \tag{6.2}$$

where $\tilde{u}(k) = \lim_{c\to\infty} d(k)$.

Note that $\tilde{c}(k), c(k)$ are the limits of the operators $\tilde{u}(k), u(k)$ respectively in the limit $\theta^{\mu\nu}=0$, and they satisfy relations (4.9). We now argue that such relations are incompatible

[8] Note that $a(k) = \lim_{\theta\to 0} d(k)$.

for $\theta^{\mu\nu} \neq 0$. Rather, $u(k)$ and $u^\dagger(k)$ fulfil certain deformed relations which reduce to (4.9) for $\theta^{\mu\nu} = 0$.

Suppose that

$$u(k)u(q) = \tilde{T}_\theta(k, q)u(q)u(k), \tag{6.3}$$

where $\tilde{T}_\theta$ is a $\mathbb{C}$-valued function of k and q yet to be determined. The transformations of $u_k u_l = (u \otimes u)(k, l)$ and $u_l u_k$ are determined by Δ_θ. Applying Δ_θ on (6.3) and using (5.10), we get the following[9]:

$$u\big(\Lambda_\infty^{-1}k\big)u\big(\Lambda_\infty^{-1}q\big)\,\mathrm{e}^{\frac{\mathrm{i}}{2}mv\theta(k_2-q_2)} = \tilde{T}_\theta(k, q)u\big(\Lambda_\infty^{-1}q\big)u\big(\Lambda_\infty^{-1}k\big)\,\mathrm{e}^{\frac{\mathrm{i}}{2}mv\theta(q_2-k_2)}. \tag{6.4}$$

Using (6.3) again on the left-hand side of (6.4), we get

$$\tilde{T}_\theta\big(\Lambda_\infty^{-1}k, \Lambda_\infty^{-1}q\big) = \tilde{T}_\theta(k, q)\,\mathrm{e}^{-\mathrm{i}mv\theta(k_2-q_2)}. \tag{6.5}$$

Note that this equation can also be obtained from the corresponding relativistic result [14] in the $c \to \infty$ limit provided one takes $\theta^{0i} = 0$ right from the beginning, otherwise the exponential factor becomes rapidly oscillating in the $c \to \infty$ limit, yielding no well-defined non-relativistic limit. Thus in the absence of spacetime noncommutativity one has an appropriate non-relativistic limit and the above-mentioned operator ordering ambiguities can be avoided.

The solution of (6.5) is[10]

$$\tilde{T}_\theta(k, q) = \eta\,\mathrm{e}^{\mathrm{i}k_i\theta^{ij}q_j} \qquad (i, j = 1, 2), \tag{6.6}$$

where η is a Galilean-invariant function and approaches the value ± 1 for bosonic and fermionic fields respectively in the limit $\theta = 0$.[11] Therefore, substituting (6.6) in (6.3) we finally have

$$u(k)u(q) = \eta\,\mathrm{e}^{\mathrm{i}k_i\theta^{ij}q_j}u(q)u(k). \tag{6.7}$$

The adjoint of (6.7) gives

$$u^\dagger(k)u^\dagger(q) = \eta\,\mathrm{e}^{\mathrm{i}k_i\theta^{ij}q_j}u^\dagger(q)u^\dagger(k). \tag{6.8}$$

Finally the creation operator $u^\dagger(q)$ carries momentum $-q$; hence, its deformed relation reads

$$u(k)u^\dagger(q) = \eta\,\mathrm{e}^{-\mathrm{i}k_i\theta^{ij}q_j}u^\dagger(q)u(k) + (2\pi)^2\delta^2(k-q). \tag{6.9}$$

Using (6.7) and (6.9), one can easily obtain the deformation algebra involving the non-relativistic fields $\psi(x)$ in the configuration space:

$$\begin{aligned} &\psi(x)\psi(y) = \int \mathrm{d}^2x'\,\mathrm{d}^2y'\,\Gamma_\theta(x, y, x', y')\psi(y')\psi(x'); \qquad \theta \neq 0 \\ &\psi(x)\psi(y) = \eta\psi(y)\psi(x); \qquad \theta = 0 \end{aligned} \tag{6.10}$$

$$\begin{aligned} &\psi(x)\psi^\dagger(y) = \int \mathrm{d}^2x'\,\mathrm{d}^2y'\,\Gamma_\theta(x, y, x', y')\psi^\dagger(y')\psi(x') + \delta^2(\vec{x}-\vec{y}); \qquad \theta \neq 0 \\ &\psi(x)\psi^\dagger(y) = \eta\psi^\dagger(y)\psi(x) + \delta^2(\vec{x}-\vec{y}); \qquad \theta = 0 \end{aligned} \tag{6.11}$$

where

$$\Gamma_\theta(x, y, x', y') = \frac{\eta}{(2\pi)^2}\exp\left(\frac{\mathrm{i}}{\theta}[(x_1'-x_1)(y_2-y_2') - (x_2'-x_2)(y_1-y_1')]\right). \tag{6.12}$$

[9] Without loss of generality, we consider the boost to be along the x^1 direction for calculational convenience. Also we set $v_1 = v$.

[10] Note that the non-relativistic form of the twist element also appears in [20].

[11] The value of η can be actually taken to be ± 1 for bosonic and fermionic fields for all $\theta^{\mu\nu}$ [14]. An exactly similar non-relativistic reduction of the Dirac equation can also be done for the fermionic case.

7. Two-particle correlation function

In this section we compute the two-particle correlation function for a free gas in 2+1 dimensions using the canonical ensemble, i.e. we are interested in the matrix elements $\frac{1}{Z}\langle r_1, r_2|\,\mathrm{e}^{-\beta H}|r_1, r_2\rangle$, where Z is the canonical partition function and H is the non-relativistic Hamiltonian. The physical meaning of this function is quite simple; it tells us what is the probability of finding particle 2 at position r_2, given that particle 1 is at r_1, i.e. it measures two-particle correlations. The relevant two-particle state is given by

$$\begin{aligned}|r_1, r_2\rangle &= \hat{\psi}^\dagger(r_1)\hat{\psi}^\dagger(r_2)|0\rangle \\ &= \int \frac{\mathrm{d}q_1}{(2\pi)^2}\frac{\mathrm{d}q_2}{(2\pi)^2} e^*_{q_1}(r_1)e^*_{q_2}(r_2)u^\dagger(q_1)u^\dagger(q_2)|0\rangle. \end{aligned} \tag{7.1}$$

The two-particle correlation function can therefore be written as

$$\langle r_1, r_2|\,\mathrm{e}^{-\beta H}|r_1, r_2\rangle = \int \mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{e}^{-\frac{\beta}{2m}(k_1^2+k_2^2)}|\langle r_1, r_2|k_1, k_2\rangle|^2, \tag{7.2}$$

where we have introduced a complete set of momentum eigenstates $|k_1, k_2\rangle$.

Using (6.9) and noting that

$$|k_1, k_2\rangle = u^\dagger(k_1)u^\dagger(k_2)|0\rangle, \tag{7.3}$$

we finally obtain

$$C(r) \equiv \frac{1}{Z}\langle r_1, r_2|\,\mathrm{e}^{-\beta H}|r_1, r_2\rangle = \frac{1}{A^2}\left(1 \pm \frac{1}{1+\frac{\theta^2}{\lambda^4}}\exp\left(-2\pi r^2\Big/\left(\lambda^2\left(1+\frac{\theta^2}{\lambda^4}\right)\right)\right)\right), \tag{7.4}$$

where A is the area of the system and λ is the mean thermal wavelength given by

$$\lambda = \left(\frac{2\pi\beta}{m}\right)^{1/2}; \qquad \beta = \frac{1}{k_B T} \tag{7.5}$$

and $r = r_1 - r_2$. The plus and the minus signs indicate bosons or fermions respectively. Although this calculation was done in 2+1 dimensions, it is clear that the result generalizes to higher dimensions by replacing θ^2 by an appropriate sum of $(\theta^{ij})^2$. The conclusions made below, based on the general structure of the correlation function, will therefore also hold in higher dimensions.

As expected this result reduces to the standard (untwisted) result in the limit $\theta \to 0$ [22]. Furthermore, it is immediately clear that when $\lambda \gg \sqrt{\theta}$, i.e. in the low temperature limit, there is virtually no deviation from the untwisted result as summarized in figure 1. This is reassuring as it indicates that the implied violation of Pauli's principle will have no observable effect at current energies. Indeed, keeping in mind that $\sqrt{\theta}$ is probably at the Planck length scale any deviation will only become apparent at very high temperatures, where the non-relativistic limit is invalidated. Note, however, that in contrast to the untwisted case the correlation function for fermions does not vanish in the limit $r \to 0$. Thus, there is a finite probability that fermions may come very close to each other. Once again this probability is determined by θ and thus very small, which probably renders it undetectable. Due to this property of the twisted correlation function one also expects that the equation of state of a free fermion gas will be much softer at high densities than the untwisted one. This is most clearly seen from the exchange potential $V(r) = -k_B T \log C(r)$ [22] shown in figure 2. This clearly demonstrates the change from a hardcore potential in the untwisted case to a soft core potential in the twisted case. This may have possible astrophysical implications, although it is dubious that these densities are even reachable in this case. In any case the assumptions we made here are certainly violated at these extreme conditions and a much more careful analysis is required to investigate the

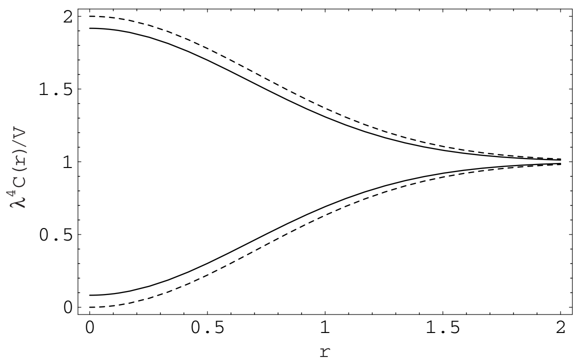

Figure 1. Two-particle correlation function $C(r)$. The upper two curves are the bosonic case and the lower curves the fermionic case. The solid line shows the twisted result and the dashed line the untwisted case. This is shown for a schematic value of $\frac{\theta}{\lambda^2} = 0.3$. The separation r is measured in units of the thermal length λ.

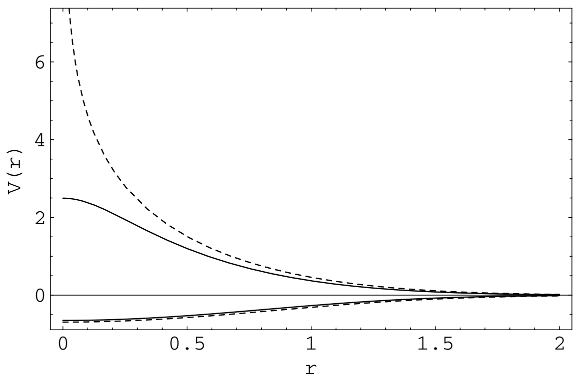

Figure 2. Exchange potential $V(r)$ measured in units of $k_B T$. The irrelevant additive constant has been set zero. The upper two curves are the fermionic case and the lower curves the bosonic case. The solid line shows the twisted result and the dashed line the untwisted case. This is shown for a schematic value of $\frac{\theta}{\lambda^2} = 0.3$. The separation r is measured in units of the thermal length λ.

high temperature and high density consequences of the twisted statistics. Another interesting point to note from figure 2 is that the twisted statistics has, even at these unrealistic values of $\frac{\theta}{\lambda^2}$, virtually no effect on the bosonic correlation function at short separation. This probably suggests that there will be no observable effect in Bose–Einstein condensation experiments. These results may also have interesting consequences for condensed matter systems such as the quantum Hall effect where the NC parameter is related to the inverse magnetic field.

8. Conclusions

We have shown that the NC parameter is twisted Galilean invariant even in the presence of spacetime noncommutativity. This is significant in view of the fact that the usual Galilean symmetry is spoiled in the presence of spacetime noncommutativity.

We have derived the deformed algebra of the Schrödinger field in configuration and momentum space. This was done by studying the action of the twisted Galilean symmetry on the Schrödinger field as obtained from a non-relativistic reduction of the Klein–Gordon field. Here we had to consider the absence of any spacetime noncommutativity as, otherwise one cannot define a proper non-relativistic limit.

The possible consequences of this deformation in terms of a violation of the Pauli principle was studied by computing the two-particle correlation function. The conclusion is that any possible effect of the twisted statistics only shows up at very high energies, while the effect at low energies should be very small, consistent with current experimental observations. Whether this effect will eventually be detectable through some very sensitive experiment is an open, but enormously interesting and challenging question.

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Appendix. A brief derivation of the Wigner-Inönu group contraction of the Poincaré group to the Galilean group

Here we summarize the well-known Wigner-Inönu group contraction from the Poincaré to Galilean algebra in order to highlight some of the subtleties involved, as these have direct bearings on the issues discussed in section 3.1.

To begin with let us consider a particle undergoing uniform acceleration 'a', along the x direction, measured in the instantaneous rest frame of the particle. A typical spacetime Rindler trajectory is given by the hyperbola

$$x^2 - c^2t^2 = \rho^2, \tag{A.1}$$

so that the acceleration $A(t)$ w.r.t. the fixed observer with the above associated coordinates (t, x) measured at time t is

$$A(t) = \frac{\mathrm{d}V(t)}{\mathrm{d}t} = \frac{c^2}{x}\left(\frac{\rho^2}{x^2}\right).$$

Since the frame (x, t) appearing in (A.1) coincides with that of the fixed observer at time $t = 0$, we must have

$$\Rightarrow \quad a = A(t = 0) = \frac{c^2}{\rho}, \tag{A.2}$$

where ρ is the distance measured at that instant from the origin. To take the non-relativistic limit, we have to take both $c \to \infty$ and $\rho \to \infty$ such that $\frac{c^2}{\rho} = a$ is held constant. For example, the corresponding non-relativistic expression $\bar{x}$ for the distance travelled by the particle in time t is obtained by identifying

$$\bar{x} = \lim_{c\to\infty,\rho\to\infty}(x - \rho) = \tfrac{1}{2}at^2, \tag{A.3}$$

which reproduces the standard result.

Now let us consider the Lorentz generator along the x direction $M_{01} = \mathrm{i}(x_0\partial_1 - x_1\partial_0)$. This can be rewritten in terms of $\bar{x}$ using (A.3),

$$\begin{aligned} M_{01} &= \mathrm{i}c\left(t\frac{\partial}{\partial\bar{x}} + \frac{1}{a}\left(1+\frac{\bar{x}}{\rho}\right)\frac{\partial}{\partial t}\right) \\ &= cK_1. \end{aligned} \tag{A.4}$$

Note that K_1 by itself does not have any c dependence; the non-relativistic limit of K_1 can thus be obtained by just taking the limit $\rho \to \infty$, which yields

$$K_1^{\mathrm{NR}} = \lim_{\rho\to\infty} K_1 = t\frac{\partial}{\partial\bar{x}} + \frac{1}{a}\frac{\partial}{\partial t}. \tag{A.5}$$

Although this vector field indeed generates the integral curve in the $t, \bar{x}$ plane which is a parabola given by (A.3), it cannot be identified with the Galileo boost generator because

$$\left[K_i^{\mathrm{NR}}, K_j^{\mathrm{NR}}\right] \sim (P_i - P_j). \tag{A.6}$$

The Galilean algebra on the other hand is obtained by taking the limit $c \to \infty$ of the commutators involving boost in the following way:

$$\begin{aligned} [\bar{K}_1, \bar{K}_2] &= \lim_{c\to\infty}\frac{1}{c^2}[M_{01}, M_{02}] = \lim_{c\to\infty}\frac{1}{c^2}M_{12} = 0 \\ [P_1, \bar{K}_1] &= \lim_{c\to\infty}\frac{1}{c}[P_1, M_{01}] = \lim_{c\to\infty}\frac{1}{c^2}P_0 = \mathrm{i}M \\ [\bar{K}_1, J] &= \lim_{c\to\infty}\frac{1}{c}[M_{01}, M_{12}] = \mathrm{i}\bar{K}_2, \end{aligned} \tag{A.7}$$

where M is identified as the mass. The rest of the commutators have the same form as that of the Poincaré algebra. This is nothing but the famous Wigner-Inönu group contraction, demonstrated here in construction of the Galilean algebra as a suitable limit of the Poincaré algebra.

A simple inspection, at this stage, shows the following form of the Galileo boost generators:

$$\bar{K}_i = K_i^{(M)} = \mathrm{i}t\frac{\partial}{\partial\bar{x}_i} + M\bar{x}_i \tag{A.8}$$

Clearly the rest of the generators in the Galilean algebra have the same form as the Poincaré algebra. For completeness we enlist the full Galilean algebra in 2+1 dimensions:

$$\begin{aligned} &\left[K_i^{(M)}, K_j^{(M)}\right] = [P_i, P_j] = [P_i, H] = [J, H] = 0 \\ &\left[P_i, K_j^{(M)}\right] = \mathrm{i}\delta_{ij}M \\ &[P_i, J] = \mathrm{i}\epsilon_{ij}P_j \\ &\left[K_i^{(M)}, J\right] = \mathrm{i}\epsilon_{ij}K_j^{(M)} \\ &[P_i, M] = [H, M] = [J, M] = \left[K_i^{(M)}, M\right] = 0. \end{aligned} \tag{A.9}$$

Finally note that here the mass M plays the role of central extension of the centrally extended Galilean algebra.

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